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APPROXIMATION OF PLURISUBHARMONIC FUNCTIONS BY MULTIPOLE GREEN FUNCTIONS

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ABSTRACT. For a strongly hyperconvex domain $D \subset \mathbb{C}^n$ we prove that multipole pluricomplex Green functions are dense in the cone in $L^1(D)$ of negative plurisubharmonic functions with zero boundary values.

1. INTRODUCTION

Pluripotential theory for studying plurisubharmonic functions is more important for complex analysis in several variables than potential theory for complex analysis in one variable because the theory in several variables lacks the multitude of methods that can be used in the classical case.

However, many beautiful results of potential theory are still either unproven or wrong in pluripotential theory. For example, the Riesz Representation Theorem states, firstly, that a subharmonic function on a nice domain D is a sum of a subharmonic function with zero boundary values and a harmonic function. Secondly, a subharmonic function u with zero boundary values is the limit of linear combinations with positive coefficients of Green functions in $L^1(D)$. So this important theorem gives a complete description of such functions.

The first statement does not hold for plurisubharmonic functions if harmonic functions are replaced by pluriharmonic. The second one immediately meets several obstacles. First of all, in potential theory a linear combination with positive coefficients of Green functions is harmonic outside of its poles. In pluripotential theory there is an analog of Green functions introduced by V. P. Zahariuta in [Z] and M. Klimek in [K1] and called *pluricomplex* Green functions. They are maximal, i.e., $(dd^c g)^n = 0$ outside of the poles, but for a sum u of pluricomplex Green functions $(dd^c u)^n$, in general, is not zero outside the poles. (The operator $(dd^c u)^n$ is an analog of the Laplace operator for plurisubharmonic functions.)

Although the absence of linear functional analysis does not allow us to expand the old methods to the new situation, in this paper we prove that multipole pluricomplex Green functions (see §2) are dense in the cone in $L^1(D)$ of negative plurisubharmonic functions with zero boundary values. Doing this, we establish the second part of the Riesz Representation Theorem.

A particular case of this problem is the approximation of the relative extremal function ω of a pluriregular compact set K (see §2). It was shown by V. P. Zahariuta

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in [Z] that the existence of approximations with uniform convergence outside K implies a Kolmogorov conjecture regarding entropies of compact sets in $\mathbb{C}^n$ (see also [Z1]). In [ZS] V. P. Zahariuta and N. P. Skiba confirmed the existence of approximations when $n = 1$. The existence of such approximations in several variables was conjectured by Zahariuta (see [Z] and [Z1]) but stayed unproven for many years. Recently A. Aytuna, A. Rashkovskii and V. P. Zahariuta proved it for pairs of Reinhardt domains (see [ARZ]).

To achieve our goal we start in §3 with relative extremal functions ω of multiple condensers. It was known from the famous paper of E. Bishop [B] that it is possible to approximate compact sets in $\mathbb{C}^n$ by analytic polyhedra defined by n holomorphic functions $f_{1j}, \dots, f_{nj}$. Since $g_j = \log\max\{|f_{1j}|, \dots, |f_{nj}|\}$ has logarithmic poles and is maximal outside of them, they are almost multipole Green functions, except for having zero values at the boundary.

It was shown by S. Nivoche in [N] that in the case of a simple condenser Bishop's construction also approximates normal derivatives of ω by normal derivatives of g_j near K. In §3 we do it for a multiple condenser.

In §4 we show that Stokes' theorem in the form of the Comparison Principle implies that the Monge–Ampère masses are also approximated. For a simple condenser this result was obtained in [N]. It was shown in [NP] that to prove Zahariuta's conjecture it suffices to construct multipole Green functions g_j with poles near K that are greater than ω near K but the total Monge–Ampère mass of g_j converges to the total Monge–Ampère mass of ω. The usage of this method in [N] led to the proof of Zahariuta's and, consequently, Kolmogorov's conjecture. We use a similar approach to prove the existence of an approximation of ω by multipole Green functions.

After that the proof of the main result in §5 follows from the possibility of approximating any continuous plurisubharmonic function uniformly by relative extremal functions of condensers and a general plurisubharmonic function by continuous plurisubharmonic functions.

2. Preliminary results

An open set $D \subset \mathbb{C}^n$ is *strongly hyperconvex* if there is a continuous plurisubharmonic function ϕ on a neighborhood V of $\overline{D}$ and $D = \{\phi < 0\}$. The function ϕ is called an *exhaustion function.*

A negative function u on D has *zero boundary values* if

$$\liminf_{z \to \partial D} u(z) = 0.$$

Suppose that $W = \{w_1, \dots, w_m\}$ is a finite set in D. We say that a plurisubharmonic function u is maximal outside W and has logarithmic poles at points of W if for every $w_j \in W$ there are a number $a_j > 0$, called the *weight* of u at w_j, and a number c such that

$$a_j \log|z - w_j| - c \le u(z) \le a_j \log|z - w_j| + c$$

near w_j, u is locally bounded on $D \setminus W$ and $(dd^c u)^n = 0$ on $D \setminus W$.

For such functions the Monge–Ampère operator can still be reasonably defined. For example, the Comparison Principle still holds (see [K2, Ch. 6]). Also (see [K2] or [D])

$$(dd^c u)^n = (2\pi)^n \sum_{j=1}^{m} a_j^n \delta_{w_j}$$

and if boundary values of u are strictly greater than c, then

$$\int_D (dd^c \max\{u, c\})^n = (2\pi)^n \sum_{j=1}^{m} a_j^n. \tag{2.1}$$

As an example of such a function, one can take any holomorphic functions $f_1, \dots, f_n$ on D such that the system $f_1 = \cdots = f_n = 0$ has simple zeros at points of W. Then the function $v = \log \max\{|f_1|, \dots, |f_n|\}$ is maximal outside W and has logarithmic poles of weight 1 at every point of W.

If D is strongly hyperconvex, then for every choice of weights a_j there is a plurisubharmonic function $g_D(z, W)$ that is continuous and maximal outside W, has logarithmic poles of weight a_j at points of W and has zero boundary values. This function is called a *multipole pluricomplex Green function.* We will always call it just the Green function.

A *pluriregular condenser* $K = (K_1, \dots, K_m, \sigma_1, \dots, \sigma_m)$ is a system of pluriregular compact sets

$$K_m \subset K_{m-1} \subset \cdots \subset K_1 \subset D \subset \overline{D} = K_0$$

and numbers $\sigma_m < \sigma_{m-1} < \cdots < \sigma_1 < \sigma_0 = 0$ such that there is a continuous plurisubharmonic function $\omega(z) = \omega(z, K, D)$ on D with zero boundary values, $K_i = \{\omega \le \sigma_i\}$ and ω is maximal on the complement of K_i in the interior of K_{i-1} for all $1 \le i \le m$. We will call this function *the relative extremal function* of the condenser K in D. Of course, not every choice of sets K_i and numbers σ_i can be realized as a condenser. But if u is a continuous negative plurisubharmonic function on D and the sets $K_i = \{u \le \sigma_i\}$ are pluriregular, then K has a continuous relative extremal function.

The ball of radius r centered at z will be denoted by $B(z, r)$. Also, $S(z, r) = \partial B(z, r)$, and $m(A)$ is the Lebesgue measure of A.

3. Approximation of condensers by holomorphic functions

This section further develops the approach of [N].

Let $K = (K_1, \dots, K_m, \sigma_1, \dots, \sigma_m)$ be a pluriregular condenser in a strongly hyperconvex open set D with an exhaustion function ϕ defined on a neighborhood V of $\overline{D}$. We assume that the sets $D^j = \{\phi < 1/j\}$ compactly belong to V for all $j = 1, 2, \dots$. Let $\omega(z) = \omega(z, K, D)$ and $D_r = \{z \in D : \omega(z) < r\}$.

Suppose that $f_1, \dots, f_N$ are holomorphic functions on some D^j, p is a positive integer and

$$v(z) = \sup_{1 \le k \le N} \frac{1}{p} \log |f_k(z)|. \tag{3.1}$$

We say that the functions $f_1, \dots, f_N$ and the integer p *approximate* K *for* $\epsilon > 0$ if for all $1 \le i \le m$ there are numbers ϵ_i, $0 < \epsilon_i \le \epsilon$, such that:

(1) $\sigma_i + \epsilon_i + \epsilon_i^2 < \sigma_{i-1}$, $1 \le i \le m$;
(2) $v(z) < \omega(z)$, $1 \le k \le N$, $z \in \overline{D}$; and
(3) if F_i, $1 \le i \le m$, is the union of all connected components of the set $\{v \le \sigma_i + \epsilon_i\}$ that intersect K_i, then $F_i \subset D_{\sigma_i + \epsilon_i + \epsilon_i^2}$.

Let G_i be the interior of F_i. Since $v < \omega$ on $\overline{D}$, $K_i \subset G_i$. Clearly, $\overline{G}_i \subset F_i$.

The set $\{v \le \sigma_i + \epsilon_i\}$ is a semianalytic set and thus has a finite number of connected components intersecting any compact set in D^j, in particular, $\overline{D}$. The set $F_i \subset D$, and therefore it has a neighborhood U where no other components of $\{v < \sigma_i + \epsilon_i\}$ are present. This means that G_i is an analytic polyhedron.

Let us show that our notion of approximation is *stable*.

Lemma 3.1. *Suppose that an integer p and holomorphic functions $f_1, ..., f_N$ on D^j approximate K for $\epsilon > 0$. Then there is a $\delta > 0$ such that for any holomorphic functions $h_1, \dots, h_N$ on D^j with the uniform norm $\|h_k\|_{\overline{D}} < \delta$ the functions $g_k = f_k + h_k$, $1 \le k \le N$, approximate K for ϵ with the same p.*

Proof. First, $|f_k| < e^{p\omega} - a$ on $\overline{D}$ for some $a > 0$ and every $1 \le k \le N$. If $\delta < a$, then $|g_k| < e^{p\omega}$.

We take δ so small that $1 - \delta e^{-p(\sigma_i+\epsilon_i)} = e^{-b_i p}$, where $0 < b_i < \epsilon_i$ for all $1 \le i \le m$. If $z \in \partial G_i$, then $|f_k(z)| = e^{p(\sigma_i+\epsilon_i)}$ for some k. Now

$$|g_k(z)| > e^{p(\sigma_i+\epsilon_i)}\left(1 - \delta e^{-p(\sigma_i+\epsilon_i)}\right) = e^{p(\sigma_i+\epsilon'_i)},$$

where $\epsilon'_i = \epsilon_i - b_i$ and $0 < \epsilon'_i < \epsilon_i$ for $1 \le i \le m$. Hence

$$v'(z) = \sup_{1\le k\le N} \frac{1}{p} \log |g_k(z)| > \sigma_i + \epsilon'_i.$$

Let G'_i be the interior of the union F'_i of connected components of the set $\{z \in D^j : v'(z) \le \sigma_i + \epsilon'_i\}$ that intersect K_i. If F' is one of the connected components of the set F'_i, then it contains a point of K_i and, consequently, intersects a connected component G of G_i. Since $v' > \sigma_i + \epsilon'_i$ on ∂G_i, the component G must contain F'. Thus $F'_i \subset G_i \subset F_i$. Also, the same reasoning implies that $G'_i \subset G_i$.

There is a positive $\epsilon''_i < \epsilon_i$ such that $F_i \subset D_{\sigma_i+\epsilon''_i+\epsilon''^2_i}$. Let us choose δ so small that $\epsilon''_i < \epsilon'_i$ for all $1 \le i \le m$. Then $F'_i \subset D_{\sigma_i+\epsilon'_i+\epsilon'^2_i}$, and the proof is complete. □

The following lemma shows the existence of holomorphic approximations.

Lemma 3.2. *For any sufficiently small $\epsilon > 0$ and any integer j there exist a positive integer p and holomorphic functions $f_1, ..., f_N$ on D^j that approximate K on $\overline{D}$ for ϵ.*

Proof. We assume that $\sigma_i + \epsilon + \epsilon^2 < \epsilon$ when $1 \le i \le m$. Let us take $0 < \delta < \min\{\epsilon^2/2, -\sigma_1\}$ and $a > 0$ so big that $a\phi < \omega$ on $\overline{D}_{-\delta}$. Then the function $\omega' = \max\{\omega, a\phi\} - \delta$ on $\overline{D}$ and $a\phi - \delta$ on $D^j \setminus D$ is plurisubharmonic on D^j. By Bremermann's approximation theorem (see [S] for a proof) there exist a positive integer p and holomorphic functions $f_1, ..., f_N$ on D^j such that

$$\omega'(z) < v(z) = \sup_{1\le k\le N} \frac{1}{p} \log |f_k(z)| < \omega'(z) + \delta$$

on D^{2j}. But $\omega' = \omega - \delta$ on $D_{-\delta}$. Hence $\omega - \delta < v < \omega$ on $D_{-\delta}$. Since $v < 0$ on ∂D and $v < -\delta$ on $\partial D_{-\delta}$, we see that $v < \omega$ on $\overline{D} \setminus D_{-\delta}$. Since $v > -\delta$ on $\overline{D} \setminus D_{-\delta}$, we see that $\omega(z) - \epsilon^2 < v(z) < \omega(z)$ on $\overline{D}$.

By the latter inequality the set $A = \{v \le \sigma_i + \epsilon\}$ belongs to $D_{\sigma_i+\epsilon+\epsilon^2}$. Hence for every $1 \le i \le m$ the union F_i of all connected components of A that intersect K_i also belongs to $D_{\sigma_i+\epsilon+\epsilon^2}$. So p and $f_1, \dots, f_N$ approximate K for ϵ. □

The following theorem approximates any condenser by n holomorphic functions.

Theorem 3.3. *For any sufficiently small $\epsilon > 0$ and any sufficiently large integer j there exist an integer p and n holomorphic functions $f_1, ..., f_n$ on D^j that approximate a pluriregular condenser K for ϵ and the system of equations $f_1 = \cdots = f_n = 0$ has only simple roots in D.*

Proof. Suppose that $N > n$ is the minimal number of holomorphic functions $f_1, \dots, f_N$ on any D^j that approximate K for ϵ with some p. Take any such approximation. By Lemma 3.1 it is stable for some $\delta > 0$. Note that none of the chosen functions is equal identically to zero. As in the proof of Lemma 4 in [B] (see also Lemma 7B1 in [GR]), we can find holomorphic functions $h_2, ..., h_N$ on D^{j+1} such that $\|h_k\|_{\overline{D}^{2j}} < \delta$ and the mapping $(f_2 f_1^{-1} + h_2, ..., f_N f_1^{-1} + h_N)$ is light, i.e., it has zero-dimensional preimages of values of every point on D^{2j}. Since $|f_1| < 1$ on $\overline{D}$, by Lemma 3.1 the functions $g_k = f_k + h_k f_1$ for $k = 2, ..., N$ and $g_1 = f_1$ approximate K for ϵ with the same p.

Let us replace the functions f_i by the functions g_i preserving notation, i.e., $f_i := g_i$. In the terminology of [B] and [GR] the analytic polyhedra G_i are now *prepared*. We want to show that for a sufficiently large integer q there is a number p' so that p' and the functions $f_k^q - f_1^q$, $2 \le k \le N$ approximate K for ϵ. The proof follows the steps of Theorem 2 in [B] or Lemma 7B2 in [GR].

There are $a < 1$ and $j_1 > j$ such that $|f_k| \le a$ on $\overline{D}^{j_1}$ for all $1 \le k \le N$. Hence $|f_k^q - f_1^q| \le 1$ on D^{j_1} when q is greater then some q_0. Also on K_i,

$$|f_k^q - f_1^q| \le 2e^{pq\sigma_i} < e^{(pq + \ln 2/\sigma_i)\sigma_i} < e^{p'\sigma_i},$$

where $p' = pq - \mu$ and μ is the least integer greater than $-\ln 2/\sigma_i$ for all $1 \le i \le m$. Note that $\mu \ge 0$. We assume that $q \ge q_1 \ge q_0$, so that $p' > 0$ for all i. Thus $|f_k^q - f_1^q| < e^{p'\omega}$ on $\overline{D}$ when $q \ge q_1$.

Let us show that $\overline{D}_{\sigma_i+\epsilon_i} \subset G_i$. Since $\sigma_i + \epsilon_i < \sigma_{i-1}$, the set $\overline{D}_{\sigma_i+\epsilon_i}$ belongs to the interior V_{i-1} of K_{i-1}. The set $\overline{G}_i$ belongs to $D_{\sigma_i+\epsilon_i+\epsilon_i^2}$, which in its turn belongs to V_{i-1} because $\sigma_i + \epsilon_i + \epsilon_i^2 < \sigma_{i-1}$. The set G_i contains K_i and, therefore, the function ω is maximal on $V_{i-1} \setminus \overline{G}_i$. The boundary of $V_{i-1} \setminus \overline{G}_i$ consists of the boundary of V_{i-1}, where $\omega = \sigma_{i-1}$, and the boundary of G_i, where $\omega > v = \sigma_i + \epsilon_i$. By the maximality of ω on $V_{i-1} \setminus \overline{G}_i$, the function $\omega > \sigma_i + \epsilon_i$ there. Hence $\overline{D}_{\sigma_i+\epsilon_i} \subset \overline{G}_i$. But $\omega > \sigma_i + \epsilon_i$ on ∂G_i and, consequently, $\overline{D}_{\sigma_i+\epsilon_i} \subset G_i$.

Hence we can find positive $\epsilon_i' < \epsilon_i$ such that

$$\overline{D}_{\sigma_i+\epsilon_i'} \subset G_i \subset \overline{G}_i \subset F_i \subset D_{\sigma_i+\epsilon_i'+\epsilon_i'^2}$$

for all $1 \le i \le m$.

Let us take an open set $U_i \subset\subset D_{\sigma_i+\epsilon_i'+\epsilon_i'^2}$ such that the domain G_i is a prepared analytic polyhedron with the frame $(U_i, g_{1i}, \dots, g_{Ni})$, where $g_{ki} = e^{-p(\sigma_i+\epsilon_i)} f_k$, i.e., $G_i = \{z \in U_i : |g_{ki}(z)| < 1, 1 \le k \le N\}$ (see [B] or [GR] for the terminology). We also take an open set $U_i' \subset\subset U_i$ containing $\overline{G}_i$. Let $V_i = U_i' \setminus \overline{D}_{\sigma_i+\epsilon_i'}$ be a neighborhood of ∂G_i. This neighborhood is relatively compact in U_i. Since $\omega = \sigma_i + \epsilon_i'$ on $\partial V_i \cap G_i = \partial D_{\sigma_i+\epsilon_i'}$ and $|f_k| < e^{p\omega}$, we see that $|g_{ki}| < r_i^{-1} = e^{p(\epsilon_i' - \epsilon_i)} < 1$ on $\partial V_i \cap G_i$ for all $1 \le k \le N$. By Theorem 2 of [B] or Lemma 7B2 from [GR], there is a sufficiently large $q_2 \ge q_1$ such that for all $q \ge q_2$ and all $1 \le i \le m$ the union R_q^i of $\overline{D}_{\sigma_i+\epsilon_i'}$ and all connected components of the set

$$\{z \in \overline{V}_i : r_i^q |g_{ki}^q - g_{Ni}^q| < 1, \, 2 \le k \le N\}$$

that intersect $\overline{D}_{\sigma_i+\epsilon'_i}$ is a polyhedron with the frame

$$(U'_i \setminus \overline{D}_{\sigma_i+\epsilon'_i}, r_i^q(g_{2i}^q - g_{1i}^q), \dots, r_i^q(g_{Ni}^q - g_{1i}^q)).$$

Recall that $p' = pq - \mu$, $\mu \geq 0$, and the function

$$v' = \sup_{2\leq k\leq N} \frac{1}{p'} \log |f_k^q - f_1^q| < \omega$$

on $\overline{D}$. It is easy to verify that $pq(\sigma_i + \epsilon'_i) > p'(\sigma_i + \epsilon''_i)$ when

$$\epsilon''_i < \epsilon'_i + \frac{\mu(\sigma_i + \epsilon'_i)}{p'} \leq \epsilon'_i.$$

We take $q_3 \geq q_2$ and ϵ''_i such that $\overline{U}'_i \subset D_{\sigma_i+\epsilon''_i+\epsilon''^2_i}$ for all $q \geq q_3$ and all i.

Let F' be a connected component of the set $F'_i = \{v' \leq \sigma_i + \epsilon''_i\}$ intersecting K_i. Let us show that $F' \subset G_i$. If $z_0 \in F' \cap K_i$, then

$$|g_{ki}^q(z_0) - g_1^q(z_0)| = e^{-pq(\sigma_i+\epsilon_i)}|f_k^q(z_0) - f_1^q(z_0)| \leq e^{-pq(\sigma_i+\epsilon_i)}e^{p'\sigma_i} < 1.$$

Thus z_0 belongs to one of the components R of the set R_q^i. If $z_1 \in \partial R$, then $|g_{ki}^q(z_1) - g_1^q(z_1)| = 1$ for some k. Hence

$$|f_k^q(z_1) - f_1^q(z_1)| = e^{pq(\sigma_i+\epsilon_i)} > e^{p'(\sigma_i+\epsilon''_i)},$$

or $v'(z_1) > \sigma_i + \epsilon''_i$. Therefore $F' \subset R \subset R_q^i$. Since $R_q^i \subset U'_i \subset D_{\sigma_i+\epsilon''_i+\epsilon''^2_i}$, we see that the functions $f_k^q - f_1^q$, $2 \leq k \leq N$, approximate K for ϵ.

So $N = n$. Let us show that the new functions can be corrected so that the system $f_1 = \dots = f_n = 0$ has only simple zeros.

Suppose that the system $f_1 = \dots = f_n = 0$ has non-simple zeros. The Jacobian of the mapping $f = (f_1, \dots, f_n)$ is not identically zero, because otherwise through every point of D there would pass a complex curve where v is constant. But $v < \sigma_1$ on K_1 and $v = \sigma_1 + \epsilon_1$ on ∂G_1.

Our approximation is stable for some $\delta > 0$. By Sard's theorem there exists a point $(c_1, \dots, c_n)$ in $\mathbb{C}^n$ such that f is non-degenerate at all preimages of this point and $|c_k| < \delta$ for all k. Let $g_i = f_i - c_i$. The system $g_1 = \dots = g_n = 0$ has only simple zeros in D, and approximates K for ϵ with the same p. ☐

4. Approximation of condensers by Green functions

The following lemma uses the existence of holomorphic approximations of relative extremal functions of pluriregular condensers to obtain an approximation of some sort of extremal functions by Green functions with the controlled behavior of Monge–Ampère masses.

Lemma 4.1. *Let $K = (K_1, \dots, K_m, \sigma_1, \dots, \sigma_m)$ be a pluriregular condenser in a strictly hyperconvex domain $D \subset \mathbb{C}^n$. Then there are sequences of positive numbers δ_j converging to zero, Green functions g_j on D, and, for every $1 \leq i \leq m$, numbers $\sigma'_{ij} < \sigma_i$ converging to σ_i and open sets V_{ij} and W_{ij} ($W_{mj} = \emptyset$) such that*

$$D_{\sigma'_{ij}} \subset\subset W_{ij} \subset\subset D_{\sigma_i} \subset\subset V_{ij} \subset\subset D_{\sigma'_{ij}},$$

$g_j > \sigma_i$ on ∂V_{ij}, $g_j > \sigma'_{ij}$ on ∂W_{ij}, the poles of g_j lie in the union of the sets $Z_{ij} = V_{ij} \setminus \overline{W}_{ij}$, $1 \le i \le m$, and

$$\int_{Z_{ij}} (dd^c\omega)^n + \delta_j \ge \int_{Z_{ij}} (dd^c g_j)^n \ge \int_{Z_{ij}} (dd^c\omega)^n - \delta_j.$$

Proof. For every $1 \le i \le m$ let us choose an increasing sequence of numbers σ'_{ij} lying strictly between σ_{i+1} and σ_i and converging to σ_i. We set $K_{2i-1,j} = K_i$ and $K_{2i,j} = \overline{D}_{\sigma'_{ij}}$. Let $\sigma_{2i-1,j} = \sigma_i$ and $\sigma_{2i,j} = \sigma'_{ij}$. We introduce the pluriregular condenser K^j formed by a system of compact sets K_{ij} and numbers σ_{ij}. Note that $\omega(z, K, D) = \omega(z, K^j, D)$ for all j.

For every j we choose a sequence of systems of holomorphic functions $f_{1j}, \dots, f_{nj}$ and integers p_j that approximate K^j for $\epsilon_j < 1/j$. We assume that the systems $f_{1j} = \dots = f_{nj} = 0$ have only simple roots and the numbers ϵ_j are so small that

$$a_j = \frac{\sigma_{ij} - \sigma_{i-1,j} + \epsilon_j}{\sigma_{ij} - \sigma_{i-1,j} + \epsilon_j + \epsilon_j^2} < 1 + \frac{1}{j}$$

for all $1 \le i \le 2m$. Let

$$v_j = \sup_{1 \le k \le n} \frac{1}{p_j} \log |f_{kj}|.$$

We will add an index j to all parameters of these approximations so that $\sigma_{ij} + \epsilon_{ij} + \epsilon_{ij}^2 < \sigma_{i-1,j}$,

$$K_{ij} \subset G_{ij} \subset \overline{G}_{ij} \subset D_{\sigma_{ij}+\epsilon_{ij}+\epsilon_{ij}^2}$$

and $v_j = \sigma_{ij} + \epsilon_{ij}$ on ∂G_{ij}.

Let ω_{ij} and v_{ij} be the auxiliary functions $(1 - \epsilon_{ij})(\omega - \sigma_{ij} - \epsilon_{ij} - \epsilon_{ij}^2)$ and $v_j - \sigma_{ij} - \epsilon_{ij}$ respectively. On ∂G_{ij} the functions $v_{ij} = 0$ and $\omega_{ij} < 0$. Since $\omega \le \sigma_{ij}$ on K_{ij} and $v_j < \omega$ on D, we see that

$$\omega_{ij} = \omega - \sigma_{ij} - \epsilon_{ij} - \epsilon_{ij}(\omega - \sigma_{ij} - \epsilon_{ij}^2) > v_{ij}$$

there. Thus the set $F_{ij} = \{v_{ij} < \omega_{ij}\} \cap G_{ij}$ contains K_{ij} and compactly belongs to G_{ij}. By the Comparison Principle,

$$\int_{G_{ij}} (dd^c v_j)^n \ge \int_{F_{ij}} (dd^c v_{ij})^n \ge \int_{F_{ij}} (dd^c \omega_{ij})^n.$$

By the maximality of ω_{ij} on $G_{ij} \setminus K_{ij}$ we get

$$\int_{G_{ij}} (dd^c v_j)^n \ge \int_{G_{ij}} (dd^c \omega_{ij})^n = (1 - \epsilon_j) \int_{G_{ij}} (dd^c\omega)^n. \tag{4.1}$$

Now we take the set P_{ij} of those poles of v_j that lie in G_{ij} and introduce the Green function g_{ij} on $D_{\sigma_{i-1,j}}$ with poles in P_{ij} of weight $1/p_j$. The functions g_{ij} have poles of the same weight as v_j in P_{ij}, and $g_{ij} = 0$ on $\partial D_{\sigma_{i-1,j}}$. Hence $g_{ij} > v_j - \sigma_{i-1,j}$ on $\overline{D}_{\sigma_{i-1,j}}$.

Let ω'_{ij} be the restriction of $a_j(\omega - \sigma_{i-1,j})$ to $D_{\sigma_{i-1,j}}$, and let $v'_{ij} = \max\{g_{ij}, \sigma_{ij} - \sigma_{i-1,j} + \epsilon_{ij}\}$. Then $\omega'_{ij} = v'_{ij} = 0$ on $\partial D_{\sigma_{i-1,j}}$ and $\omega'_{ij} < v'_{ij}$ on G_{ij}, because $\omega'_{ij} < \sigma_{ij} - \sigma_{i-1,j} + \epsilon_{ij}$ there. Since the function $g_{ij} > \sigma_{ij} - \sigma_{i-1,j} + \epsilon_{ij}$ on ∂G_{ij} and is maximal on $D_{\sigma_{i-1,j}} \setminus \overline{G}_{ij}$, we see that $\omega_{ij} \le v_{ij}$ on $D_{\sigma_{i-1,j}}$. By the Comparison Principle,

$$\int_{D_{\sigma_{i-1,j}}} (dd^c v'_{ij})^n \le a_j \int_{D_{\sigma_{i-1,j}}} (dd^c \omega)^n.$$

By (2.1),

$$\int_{D_{\sigma_{i-1,j}}} (dd^c v'_{ij})^n = \int_{D_{\sigma_{i-1,j}}} (dd^c g_{ij})^n.$$

By the maximality of g_{ij} and ω on $D_{\sigma_{i-1,j}} \setminus \overline{G}_{ij}$ we get

$$\int_{G_{ij}} (dd^c v_j)^n = \int_{G_{ij}} (dd^c g_{ij})^n \le a_j \int_{G_{ij}} (dd^c \omega)^n. \tag{4.2}$$

Since

$$\int_{G_{2i,j}} (dd^c \omega)^n = \int_{G_{2i+1,j}} (dd^c \omega)^n,$$

by (4.1) and (4.2)

$$\int_{G_{2i,j}} (dd^c v_j)^n - \int_{G_{2i+1,j}} (dd^c v_j)^n \le \frac{2}{j} \int_{G_{2i+1,j}} (dd^c \omega)^n. \tag{4.3}$$

Now for $1 \le i \le m$ we let $V_{ij} = G_{2i-1,j}$ and $W_{ij} = G_{2i,j}$. For each j let us consider the Green function g_j on D with poles of weight $1/p_j$ at those poles of v_j that lie in the union of the sets G_{2m-1} and $G_{2i-1,j} \setminus \overline{G_{2i,j}}$, $1 \le i \le m-1$. By the definition of g_j and (4.2),

$$\int_{V_{ij}} (dd^c g_j)^n \le \int_{V_{ij}} (dd^c v_j)^n \le \left(1 + \frac{1}{j}\right) \int_{V_{ij}} (dd^c \omega)^n.$$

Now

$$\int_{V_{ij}} (dd^c g_j)^n = \int_{V_{ij}} (dd^c v_j)^n - \sum_{k=i}^{m-1} \int_{G_{2k,j} \setminus \overline{G}_{2k+1,j}} (dd^c v_j)^n.$$

By (4.1) and (4.3),

$$\int_{V_{ij}} (dd^c g_j)^n \ge \left(1 - \frac{2m}{j}\right) \int_{V_{ij}} (dd^c \omega)^n.$$

Let $\rho_j = 2m/j$ and

$$\delta_j = 2\rho_j \int_D (dd^c \omega)^n.$$

By the above inequalities,

$$\begin{aligned} \int_{Z_{ij}} (dd^c g_j)^n &= \int_{V_{ij}} (dd^c g_j)^n - \int_{V_{i+1,j}} (dd^c g_j)^n \\ &\le (1 + \delta_j) \int_{V_{ij}} (dd^c \omega)^n - (1 - \delta_j) \int_{V_{i+1,j}} (dd^c \omega)^n \le \int_{Z_{ij}} (dd^c \omega)^n + \delta_j. \end{aligned}$$

In the same way,

$$\int_{Z_{ij}} (dd^c g_j)^n \geq (1-\delta_j) \int_{V_{ij}} (dd^c \omega)^n - (1+\delta_j) \int_{V_{i+1,j}} (dd^c \omega)^n$$
$$\geq \int_{Z_{ij}} (dd^c \omega)^n - \delta_j.$$

Since $g_j > v_j$ on D, we see that $g_j > \sigma_i$ on ∂V_{ij} and $g_j > \sigma'_{ij}$ on ∂W_{ij}. The lemma is proved. $\square$

The following lemma shows that a sequence of Green functions obtained in the previous result approximates ω in the more usual sense. For a simple pluriregular condenser the lemma was proved in [NP].

Lemma 4.2. *Suppose that a sequence of Green functions g_j on D satisfies the conditions of Lemma 4.1. Then this sequence converges to $\omega(z) = \omega(z, K, D)$ uniformly on compacta in $D_{\sigma_{i-1}} \setminus K_i$, $1 \leq i \leq m$. Moreover, if ψ is a continuous function on $\mathbb{R}$, then*

$$\lim_{j\to\infty} \int_D \psi(\omega(z))(dd^c g_j)^n = \int_D \psi(\omega(z))(dd^c \omega)^n.$$

Proof. Let us denote by g_{ij} the restriction of g_j to the open set Z_{ij}. Since $g_j > \sigma_i > \sigma'_{ij}$ on ∂V_{ij} and $g_j > \sigma'_{ij}$ on ∂W_{ij}, the set $\{g_{ij} < \sigma'_{ij} - \delta_j\}$ compactly belongs to Z_{ij}. Let $g'_{ij} = \max\{g_{ij}, \sigma'_{ij} - \delta_j\}$. Then the function g'_j defined as g'_{ij} on Z_{ij}, $1 \leq i \leq m$, and g_j otherwise is plurisubharmonic on D. Moreover, by (2.1),

$$\int_{Z_{ij}} (dd^c g'_j)^n = \int_{Z_{ij}} (dd^c g_j)^n.$$

Let

$$\alpha_j = \min\left\{ \frac{\sigma'_{ij}}{\sigma'_{ij} - \delta_j} : 1 \leq i \leq m \right\}.$$

Clearly, α_j converges to 1.

Let $v_j = \alpha_j g'_j$. Since $v_j \geq \sigma'_{ij} > \omega$ on every Z_{ij}, by the maximality of v outside of the union of the Z_{ij} it is greater than ω on D. Therefore, integrating by parts, we get

$$\int_D (-\omega)(dd^c \omega)^n \geq \int_D (-v_j)(dd^c \omega)^n = \int_D (-\omega) dd^c v_j \wedge (dd^c \omega)^{n-1}$$
$$\geq \int_D (-v_j) dd^c v_j \wedge (dd^c \omega)^{n-1} = \cdots \geq \int_D (-v_j)(dd^c v_j)^n.$$

Thus

$$0 \leq \int_D (v_j - \omega)(dd^c \omega)^n \leq \int_D v_j (dd^c v_j)^n - \int_D \omega (dd^c \omega)^n$$
$$= \sum_{i=1}^m \left(\int_{Z_{ij}} v_j (dd^c v_j)^n - \int_{Z_{ij}} \omega (dd^c \omega)^n \right).$$

The support of $(dd^c\omega)^n$ lies in the boundaries of the K_i, where $\omega = \sigma_i$. The support of $(dd^c v_j')^n$ lies where $g_j' = \sigma_{ij}' - \delta_j$. Thus, by the previous inequality,

$$0 \le \int_D (v_j - \omega)(dd^c\omega)^n \le \sum_{i=1}^m \left(\alpha_j^n \sigma_{ij}' \int_{Z_{ij}} (dd^c g_j')^n - \sigma_i \int_{Z_{ij}} (dd^c\omega)^n \right).$$

By (2.1) and the integral inequality in Lemma 4.1,

$$\int_{Z_{ij}} (dd^c g_j')^n = \int_{Z_{ij}} (dd^c g_j)^n \ge \int_{Z_{ij}} (dd^c\omega)^n - \delta_j.$$

Hence

$$0 \le \int_D (v_j - \omega)(dd^c\omega)^n \le \sum_{i=1}^m \left((\alpha_j^n \sigma_{ij}' - \sigma_i) \int_{Z_{ij}} (dd^c\omega)^n - \delta_j \alpha_j^n \sigma_{ij}' \right).$$

Therefore,

$$\lim_{j\to\infty} \int_{\{v_j - \omega > a\}} (dd^c\omega)^n = 0$$

for every $a > 0$.

We fix some $\delta > 0$ and choose $\epsilon > 0$ such that $\epsilon|z|^2 < \delta/2$ on D, and we let $u_j = v_j + \epsilon|z|^2 - \delta$.

Note that

$$\left(dd^c(\epsilon|z|^2 - \delta)\right)^n = \epsilon^n c_n \, dV,$$

where the constant c_n depends only on n, and dV is the volume form. Let $E_j = \{z \in D : \omega < u_j\}$. Since $u_j < -\delta/2$ on ∂D and $v_j - \omega > \delta/2$ on E_j, the set E_j compactly belongs to D and $E_j \subset \{v_j - \omega > \delta/2\}$. By the subadditivity of the Monge–Ampère operator and the Comparison Principle we have

$$\int_{E_j} (dd^c v_j)^n + \int_{E_j} (dd^c \epsilon|z|^2 - \delta)^n \le \int_{E_j} (dd^c u_j)^n \le \int_{E_j} (dd^c\omega)^n$$

or

$$\epsilon^n c_n m(E_j) \le \int_{\{v_j - \omega > \delta/2\}} (dd^c\omega)^n.$$

Thus

$$\lim_{j\to\infty} m(E_j) = 0.$$

The set $F_j = \{\omega < v_j - \delta\} \subset E_j$. Let us take $r > 0$ such that $|\omega(z) - \omega(w)| < \delta$ when $|z - w| < r$, and take j_0 such that $m(E_j) < \delta m(B(z, r))$ for all $j \ge j_0$. If $B = B(z_0, r) \in D$, then

$$v_j(z_0) \le \frac{1}{m(B)} \int_B v_j(z) \, dV = \frac{1}{m(B)} \left(\int_{B\setminus F_j} v_j \, dV + \int_{B\cap F_j} v_j \, dV \right)$$
$$\le \frac{1}{m(B)} \left(\int_B (\omega + \delta) \, dV - \int_{B\cap F_j} (\omega + \delta) \, dV \right) \le \omega(z_0) + (2 - \sigma_m)\delta.$$

Since $v_j(z) \geq \omega(z)$, we see that the functions v_j converge to ω uniformly on D. Consequently, the functions g_j converge to ω uniformly on compacta in $D_{\sigma_{i-1}} \setminus K_i$, $1 \leq i \leq m$.

The last statement of the lemma follows immediately from the integral inequality in Lemma 4.1. □

5. Approximation of plurisubharmonic functions

Theorem 5.1. *If D is a strongly hyperconvex domain in $\mathbb{C}^n$ with an exhaustion function ϕ and u is a negative plurisubharmonic function on D with zero boundary values, then there is a sequence of pluricomplex multipole Green functions g_j converging to u in $L^1(D)$. Moreover, if u is continuous on D and a function $\psi \in C_0((-\infty, 0])$, then*

$$\lim_{j\to\infty} \int_D \psi(u(z))(dd^c g_j)^n = \int_D \psi(u(z))(dd^c u)^n.$$

Proof. We start to prove this theorem for a continuous plurisubharmonic function u on D with zero boundary values, for which there is an open set $D' \subset\subset D$ such that $\partial D'$ is a smooth hypersurface, u is equal to $\sigma_1 < 0$ on $\partial D'$, maximal on $D \setminus \overline{D}'$ and is of class C^2 and strictly plurisubharmonic in D'. Then u has finitely many local minima $z_1, \dots, z_p$ in D'.

By Sard's theorem, for every $j \geq 1$ we can find numbers $\sigma_{m_j j} < \sigma_{m_j -1,j} < \cdots < \sigma_{2j} < \sigma_{1j} = \sigma_1 < \sigma_0 = 0$ such that:

(1) $\sigma_{ij} - \sigma_{i+1,j} < 1/j$ and the function u is not degenerate on $\{u = \sigma_{ij}\}$, $1 \leq i \leq m_j$;
(2) if z_k is a local minimum of u and $\sigma_{ij} < u(z_k) < \sigma_{i-1,j}$, then the connected component of the set $\{u < \sigma_{i-1,j}\}$ that contains z_k belongs to a ball B_{kj} and $\sum_{k=1}^p m(B_{kj}) < 1/j$.

By the first condition the sets $K_{ij} = \{u \leq \sigma_{ij}\}$ have smooth boundaries and, therefore, are pluriregular. So the condenser K_j formed by the K_{ij} and σ_{ij}, $0 \leq i \leq m_j$, has the continuous relative extremal function ω_j.

For every ω_j we take a sequence of Green functions g_{jm} provided by Lemma 4.2 and select a subsequence g_{jq_m} that converges in $L^1_{loc}(D)$ to a plurisubharmonic function v_j or $-\infty$. But it cannot converge to $-\infty$, and, because it converges to ω_j uniformly on compacta in $\overline{D} \setminus \overline{D}'$, it converges to v_j in $L^1(D)$. For every i the function v_j coincides with ω_j on the sets $\{\omega_j < \sigma_{i-1,j}\} \setminus K_{ij}$ or everywhere except the sets $\{\omega_j = \sigma_{ij}\}$. A set $\{\omega_j = \sigma_{ij}\}$ either is a smooth surface or contains a local minimum z_k. In the first case $v_j = \omega_j$ on this set. In the second case the set belongs to the ball B_{kj}.

Let A be the infimum of u on D. There is a point in a ball B_{kj} where $v_j(z) = \omega_j(z) \geq A$. If B'_{kj} is a ball centered at that point and of radius twice the radius of B_{kj}, then

$$\sum_{k=1}^p \int_{B'_{kj}} v_j \, dV \geq \sum_{k=1}^p Am(B'_{kj}) \geq \frac{A}{j}.$$

Thus

$$\int_D |v_j - \omega_j| \, dV \leq \frac{2A}{j}.$$

Since the functions ω_j converge uniformly to u, the functions v_j converge to u in $L^1(D)$ and the functions $(dd^c\omega_j)^n$ weak-$*$ converge to $(dd^cu)^n$.

Let us take a countable dense set of functions $\{\psi_q\}$ in $C_0((-\infty,0])$. Since there is a sequence of g_{jp_j} that converges to u in $L^1(D)$ and

$$\lim_{j\to\infty}\int\limits_D \psi_q(u(z))(dd^cg_{jp_j})^n = \int\limits_D \psi_q(u(z))(dd^cu)^n$$

for every q, our theorem is proved for the functions of the special type.

Suppose that ϕ is defined on a neighborhood V of $\overline{D}$. Let u be a continuous plurisubharmonic function on D with zero boundary values. The sequence of plurisubharmonic functions u_k equal to $\max\{u, k\phi\}$ on D and $k\phi$ on $V\setminus D$ is decreasing on D and converges to u uniformly on $\overline{D}$. In particular, $(dd^cu_k)^n$ weak-$*$ converges to $(dd^cu)^n$. Hence to prove our theorem for continuous functions it suffices to prove it for continuous functions that admit a continuous plurisubharmonic extension to V.

If u is such a function, then there is a decreasing sequence of plurisubharmonic functions u_k on some D^j that belong to $C^\infty(D^j)$ (see [K2, Theorem 2.9.2]) and converge to u uniformly on $\overline{D}$. Adding $\epsilon_k|z|^2-\delta_k$ to u_k, where the numbers $\epsilon_k,\delta_k>0$ converge to 0 and $\epsilon_k|z|^2-\delta_k<0$ on D, we may assume that the functions u_k are strictly plurisubharmonic, $u_k<0$ on $\overline{D}$, and they still converge uniformly to u on $\overline{D}$.

Let us choose a sequence of numbers $\sigma_{1k}<0$ converging to 0 such that for all k the set $\{u_k=\sigma_{1k}\}$ is a smooth hypersurface compactly belonging to D. We define u_k' as a plurisubharmonic function that is equal to u_k on the set $W_k=\{u_k\le\sigma_{1k}\}$, to 0 on ∂D, and is maximal on $D\setminus W_k$. These functions converge uniformly to u. Since for functions like this our theorem is already proved, it is proved for continuous plurisubharmonic functions.

Any plurisubharmonic function u on D belongs to $L^1_{loc}(D)$, and if u has zero boundary values, then it belongs to $L^1(D)$. The functions u_j that are equal to $\max\{u, j\phi\}$, $j=1,2,\dots$, on D and to $j\phi$ outside D form a decreasing sequence of functions on D converging to u. Hence they converge to u in $L^1(D)$. Using [K2, Theorem 2.9.2] again, we approximate each u_j by a decreasing sequence $\{u_{kj}\}$ of continuous plurisubharmonic functions. Since $u_j=0$ on ∂D, it is easy to see that the functions u_{kj} can be modified to be equal to 0 on ∂D and still converge to u_j in $L^1(D)$. The result above implies the existence of Green functions converging to u_j and, consequently, to u in $L^1(D)$. □

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